

# TOROIDAL SPIN-NETWORKS: TOWARDS A GENERALIZATION OF THE DECOMPOSITION THEOREM

HANS-CHRISTIAN RUIZ

Max-Planck-Institute for Gravitational Physics, Golm, Germany (Internship)

Email: *hans.christian.ruiz@aei.mpg.de*

**ABSTRACT.** In this paper Moussouris' algorithm for the decomposition of spin networks is reviewed and the implicit assumptions made in the Decomposition Theorem relating a spin network with its state sum are examined. It is found that the theorem in the original form hides the importance of the orientation of the vertices in the spin networks and, more important, that the algorithm for the evaluation of spin networks assumes a cycle for the reduction of the graph to work in the proof of the theorem. It is shown that this is not always the case and this rises doubt about the generality of the theorem. Having this issue in mind, the theorem is restated to account for toroidal spin networks, i.e. networks cellular embeddable in the torus. For this, the minimal non-planar spin network is examined and the algorithm is extended to account for the non-planarity of toroidal spin networks. Furthermore, three types of minimal non-planar spin networks are found, two of them are toroidal and the third is cellular embeddable only in the double torus and without a cycle. Some examples for both toroidal spin networks are given and the relation between these is found. Finally, the issues and the possibility of generalizing the Decomposition Theorem is shortly discussed.

## 1. INTRODUCTION

Moussouris' algorithm for the evaluation of planar spin networks and the Decomposition Theorem are important tools for the evaluation of spin networks, for instance one uses them in order to extract the state sum of a triangulated 3-dimensional manifold with boundary giving rise to the Ponzano-Regge theory relating the 1-skeleton of the triangulation to the "partition function" of the geometry of the given manifold. This is made by considering the triangulation of the boundary as the dual to a spin network and decomposing the manifold, by means of Moussouris' algorithm, into tetrahedra, which are related in a specific way to the  $6j$ -symbols found in the coupling theory of  $SU(2)$ -representations. This association allows the construction of the state sum of the given manifold as a sum over a product of  $6j$ -symbols, [Moussouris(1983), Ponzano and Regge(1969), Regge(1961)]. This idea can be generalized to a more abstract setting as described in [Barrett and Westbury(1996)], for instance the model in [Turaev and Viro(1992)], where the category of representations of the quantum group  $U_q(\mathfrak{sl}_2)$  is used, rather than the category of  $SU(2)$ -representations as in the Ponzano-Regge model.

When considering general spin networks, one has to rise the question about the global topology of the manifold encoded in the spin network under consideration and whether is possible to blindly apply the Decomposition Theorem to obtain a state sum encoding this information. In other words, how is the global topology of the manifold reflected in the evaluation of the corresponding spin network? This question is important since from the graph-theoretical point of view planar spin networks, i.e. those embeddable in a sphere, are a special case and one has to imagine more complicated generic spin networks if one takes them to be quantum states of 3-dimensional space. In what follows we will explore this issue using some basic concepts of (topological) graph theory taken from [Hartsfield and Ringel(2003), Beineke and Wilson(1997)] to describe the embeddings of spin networks in surfaces, in particular the cellular embeddings of non-planar graphs with a  $(3,3)$ -bipartite graph as subgraph, and present an extension of Moussouris' algorithm in order to account for the non-planarity of the minimal spin network with non-trivial topology, i.e. non-trivial topology of the surface in which it is cellular embeddable.

The reader familiarized with graph theory and the evaluation of spin networks via Moussouris' algorithm might want to take a look at the example 3 on page 5 and the following ones and jump directly to 3.2 on page 11. However, to make this paper self-contained we will give some basic notions of graph theory with especial attention to Kuratowski's Theorem stating an easy condition for a graph to be non-planar, and

the Rotation Scheme Theorem describing how to obtain all cellular embeddings for a given graph out of a simple rule. Furthermore, we will consider the  $(3,3)$ -bipartite graph  $K_{3,3}$  as the minimal non-planar spin network, construct its non-equivalent cellular embeddings and show that these are the only possible ones up to permutation of its vertices and their relative orientation among those vertices of the same set in  $K_{3,3}$ .

Moreover, Moussouris' algorithm for the evaluation of planar spin networks and the Decomposition Theorem are presented in section 3.1 on page 8 of this paper. We will apply and extend these ideas to evaluate the above mentioned spin networks, which will allow us to define a toroidal symbol in order to attempt a generalization of the Decomposition Theorem. To achieve this a few identities for relating the evaluation of the different embeddings in the torus are given. We explain the main concepts involving the generalization of the evaluation of non-planar spin networks in terms of toroidal symbols, however, the process for obtaining the main result is still ongoing, thus some caveat concerning the generalization of Kuratowski's Theorem and its possible solution will be pointed out.

## 2. KURATOWSKI'S THEOREM AND THE EMBEDDING OF GRAPHS IN SURFACES

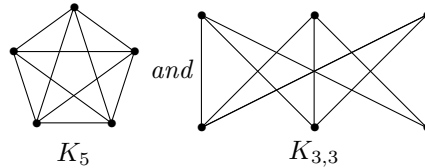
A **graph**  $G$  is a pair of sets  $(V_G; E_G)$  where  $V_G \neq \emptyset$  is the set of vertices of  $G$  and  $E_G$  is a set of unordered pairs of elements of  $V_G$  which might be empty. The pairs of vertices are called edges and they are defined, in an abstract way, as a relation between the vertices in  $G$ . If two vertices form an edge, we say they are adjacent or neighbors. A **subgraph** of a graph  $G$  is a graph  $H = (V_H; E_H)$  such that  $V_H \subseteq V_G$  and  $E_H \subseteq E_G$ , [Hartsfield and Ringel(2003)]. Another concept related to subgraphs are the **minors** of a graph; these are graphs obtained from  $G$  by a succession of edge-deletions and edge-contractions. If the minor  $M$  was obtained only by edge contractions, then  $G$  is said to be contractible to  $M$ , [Beineke and Wilson(1997)].

We are able to consider only trivalent vertices since spin networks of higher degree, i.e. with vertices of higher valence, are expandable to cubic graphs. On the other hand, a spin network is only allowed to have at most one edge between two adjacent vertices. If two vertices are joined by more than one edge, the structure is called a multigraph. In the case of cubic graphs we only have trivalent vertices, hence, a multigraph would have two vertices joined by at most three edges. This impose, however, no constraints in the class of spin networks since the double edge can be reduced to a single edge using Schur's lemma and a triple edge is exactly the theta-net defined as the value of a 3-vertex.

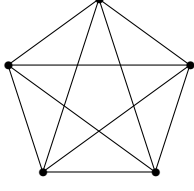
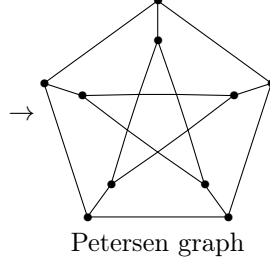
Two graphs are **homeomorphic** if they can be obtained from the same graph by subdividing its edges. Subdividing an edge  $e = vw$  between two vertices  $v, w$  is the operation of inserting a new vertex  $z$  such that  $e$  is replaced by two new edges  $vz$  and  $zw$ , [Beineke and Wilson(1997)].

An embedding of a graph on a surface is called **cellular** if each region is homeomorphic to an open disc. It is in this sense that planarity is defined, namely, a graph is planar if it can be embedded in a plane, hence, in the 2-sphere. Surprisingly, there is a simple criterion for determining whether a graph is planar, or not, given by Kuratowski's theorem.

**Theorem 1. Kuratowski's Theorem.** *A graph is planar if and only if it has neither  $K_5$  nor  $K_{3,3}$  as a minor, i.e. there are no subgraphs homeomorphic or contractible to  $K_5$  and  $K_{3,3}$ .*



The graph  $K_5$  is called the complete graph on five vertices and  $K_{3,3}$  is the  $(3,3)$ -bipartite graph. Notice that the  $K_5$  graph is 4-valent and can be expanded as a spin network to obtain the Petersen graph which is a cubic graph with 10 vertices as depicted below. If one deletes any of its vertices and the edges incident to it, one finds a graph homeomorphic to the  $(3,3)$ -bipartite graph. Hence, we only need to focus on the latter graph.

 $K_5$ 

Petersen graph

Closed surfaces, on the other hand, are categorized into orientable and non-orientable surfaces, e.g. the sphere  $S^2$ , torus  $T^2$  or the real projective plane  $P^2$ . Any oriented surface is homeomorphic to the sphere or to the connected sum  $T^2 \# T^2 \# \dots \# T^2$  of a finite number  $h$  of tori. Here we will only consider the embeddings of spin networks in orientable closed surfaces.

Whether a given graph  $G$  is embeddable in an orientable surface  $S_h$  or not depends on its genus  $\gamma(G)$ , which is defined to be the minimum genus of any orientable surface in which  $G$  is embeddable, i.e.  $G$  is embeddable in  $S_h$  if  $h \geq \gamma(G)$ . In fact, any graph can be embedded in a surface with enough handles just by adding a handle at each crossing, but we are rather interested in cellular embeddings<sup>1</sup> for which Euler's formula hold,

$$(2.1) \quad v - e + f = 2 - 2h$$

where  $v, e, f$  are the number of vertices, edges and faces respectively and  $h$  is the (orientable) genus of  $S_h$ . In this context, a planar graph has genus  $\gamma(G_{planar}) = 0$ .

There is no general formula for calculating the orientable genus of a given graph, however, for the  $(s, r)$ -bipartite graph it is given by

$$(2.2) \quad \gamma(K_{s,r}) = \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil$$

where  $\lceil x \rceil$  denotes the next integer bigger than  $x$ . Hence,  $K_{3,3}$  is embeddable in the torus but not in the sphere since  $\gamma(K_{3,3}) = \lceil \frac{1}{4} \rceil = 1$ , [Beineke and Wilson(1997)].

Notice that from the equation (2.1) there is a topological constraint to the allowed cellular embeddings for a given graph. For instance,  $K_{3,3}$  has 6 vertices and 9 edges, thus, we obtain a constraint for the number of faces,  $f = 5 - 2h$ , since  $f, h > 0$ . Furthermore, from equation (2.2) we have  $h \geq 1$ , hence,  $f = 3$  or  $f = 1$  in the case where  $K_{3,3}$  is embedded<sup>2</sup> in  $T^2$  or  $T^2 \# T^2$  respectively.

Is there other information encoded in the graph that can help us to obtain all possible embeddings? Does the orientation of the vertices impose a constraint on the embedding? Now, having found the number of possible faces (or 2-cells) for the embeddings, we want to construct oriented surfaces such that the cellular embeddings are automatically realized. In order to achieve this, we need to find the circuits of the given graph  $G$  and attach 2-cells to the regions bounded by them.

A **circuit** is a closed walk<sup>3</sup> in  $G$  such that no edge is repeated in the same direction. One may imagine a circuit as walking along an edge in a certain direction and, when getting to a vertex, the direction to

<sup>1</sup>In fact, if  $\gamma(G) = h$ , then every embedding of  $G$  on  $S_h$  is cellular, [Beineke and Wilson(1997)].

<sup>2</sup>From now on, whenever we refer to an embedding, it is meant a cellular embedding.

<sup>3</sup>A walk in  $G$  is an alternating sequence  $v_1 e_1 v_2 e_2 \dots v_{n-1} e_{n-1} v_n$  of vertices and edges of  $G$ , where every edge  $e_i$  is incident with  $v_i$  and  $v_{i+1}$ , and  $v_i \neq v_{i+1}$ . If  $v_1 = v_n$ , then it is a closed walk, [Hartsfield and Ringel(2003)]. It is important to notice here that this definition is a property of the graph itself. We will, however, abuse the use of the language and refer to the regions bounded by the circuits (cf. theorem 2) as *embedded circuits*, when they have repeated edges in both directions, or as *embedded cycles* when no edges are repeated. To clarify, the difference is that the latter concepts are related to the embedding of the graph in a surface and the definition given above is a property of the graph related only indirectly to the embedding of the graph through theorem 2.

follow (either clockwise or counter-clockwise) is given by the orientation, also called rotation, of the vertex under consideration. Thus, a given configuration of the orientations of all vertices in the graph, called a rotation scheme, induces a set of circuits giving rise to a specific cellular embedding. Hence, all embeddings can also be described by giving the orientation<sup>4</sup> of each vertex and specifying the regions bounded by the circuits obtained from applying the so called **rotation rule**: after the edge  $xy$ , take the edge  $yz$ , where  $z$  is the successor to  $x$  in the permutation of the neighbors of the vertex  $y$ , see examples below. The previous discussion is formalized in the next theorem, [Beineke and Wilson(1997)].

**Theorem 2. Rotation Scheme Theorem.** *Let  $G$  be a connected graph with  $v$  vertices and  $e$  edges, and let  $\Pi = \{\pi_1, \pi_2, \dots, \pi_v\}$  be a set of cyclic permutations of the neighbors of the vertices  $\{1, 2, \dots, v\}$ , i.e. a set of a given orientation of all vertices. Let  $W_1, W_2, \dots, W_f$  be the circuits obtained by applying the rotation rule to  $\Pi$ . Then the circuits are the boundaries of the regions of a cellular embedding of  $G$  in  $S_h$ , with  $h = (2 - v + e - f)/2$ , the genus of the orientable surface  $S_h$ . Hence, all possible cellular embeddings of a graph are provided by the rotation schemes.*

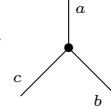
Let us consider the graph  $K_{3,3}$  with  $v = 6$ ,  $e = 9$ . This is a graph with trivalent vertices, hence, there are two possible rotations<sup>5</sup> for each of the six vertices. As a consequence, there are  $2^6 = 64$  different sets  $\Pi_{i=1, \dots, 64}$ . The question that arises immediately is whether all these sets induce topological inequivalent embeddings or not. In other words, if we disregard the labeling of the vertices, how many different embeddings of  $K_{3,3}$  in the torus or the double-torus exist?

Before making a general claim, let us work out some examples to illustrate the construction of embeddings by the rotation rule in order to understand the relation between the set of orientations and the embeddings induced by them. We will achieve this by listing the vertices and their neighbors and from this list extract the circuits in the embedding using the rotation rule, [Beineke and Wilson(1997), Hartsfield and Ringel(2003)].

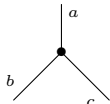
Notice that for a general  $(r, s)$ -bipartite graph the defining characteristics are: (i) The  $r$  vertices corresponding to a set, say  $R$ , are adjacent to each of the  $s$  vertices corresponding to another set, say  $S$ , and (ii)  $R \cap S = \emptyset$ . In the case of  $K_{3,3}$  each of the odd vertices,  $R = \{1, 3, 5\}$ , are adjacent to each of the even vertices,  $S = \{2, 4, 6\}$ . Thus, we denote (up to cyclic permutations) positive orientations of even and odd vertices as (135) and (246) respectively, and (153) and (264) for negative orientations. For instance, a standard way of picturing  $K_{3,3}$  with minimal crossing is

<sup>4</sup>The orientation is given as a cyclic permutation of the neighbors encountered while going clockwise around the vertices. In this convention, the orientation of a vertex  $v$  is the equivalent class of even permutations, in the case of positive orientation denoted  $(+1)$ , or odd permutations, in the case of negative orientation denoted  $(-1)$ , of its neighbors. Such a definition takes into account the fact that both directions, anticlockwise and clockwise, can be described by listing the neighbors in order of their

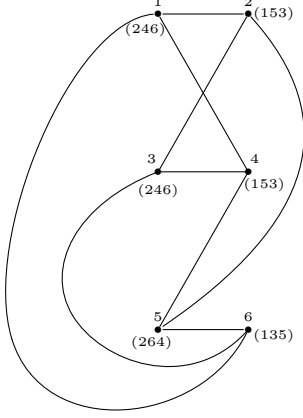
appearance when going *clockwise*, for instance, if  $(abc)$  describes the positive orientation of



negative orientation, meaning going around counter-clockwise, which can also be regarded as permutating the edges  $vc$  and  $vb$

giving . Notice that *in the diagram* the listing of neighbors of  $v$  is still clockwise.

<sup>5</sup>In fact, if a vertex  $v$  has degree  $d$ , then there are  $(d - 1)!$  different rotations of  $v$ .



This configuration has orientations (246) for the vertices 1 and 3, (264) for vertex 2 and (153) for the vertices 2 and 4, (135) for vertex 6.

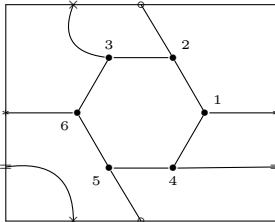
**Example 3.** Consider the case where the vertices in each set have the same orientation, i.e. either  $(+1, +1, +1)$  or  $(-1, -1, -1)$ . For instance, the vertices  $\{1, 3, 5\}$  have negative orientation while the vertices  $\{2, 4, 6\}$  have all positive orientation:

Vertex	1	2	3	4	5	6
Neighbors/Orientation	(264)	(135)	(264)	(135)	(264)	(135)

From this information we extract the circuits which will help us to construct the embedding corresponding to this configuration. For instance, take the edge (12) and apply the rotation rule on it, i.e. the neighbor of 2 coming after 1 in the cyclic permutation (135) is 3, hence, the next edge in the walk is (23). Apply again the rule to get (36) and so on. After some steps, depending on how long the walk is, one gets to the edge where the procedure started, meaning that one has to stop and apply the same procedure to another edge different than the ones encountered in the previous walk. In this specific configuration, this algorithm results in the following disjoint circuits,

- (1) (12)  $\rightarrow$  (23)  $\rightarrow$  (36)  $\rightarrow$  (65)  $\rightarrow$  (54)  $\rightarrow$  (41)  $\rightarrow$  (12);
- (2) (25)  $\rightarrow$  (56)  $\rightarrow$  (61)  $\rightarrow$  (14)  $\rightarrow$  (43)  $\rightarrow$  (32)  $\rightarrow$  (25);
- (3) (21)  $\rightarrow$  (16)  $\rightarrow$  (63)  $\rightarrow$  (34)  $\rightarrow$  (45)  $\rightarrow$  (52)  $\rightarrow$  (21).

Notice that there is a difference between the “side”  $(xy)$  and  $(yx)$  reflecting the direction of the walk, hence, there are 18 “sides” available to build the circuits. In this case, there are three regions with six sides as boundaries, thus, the embedding has three faces and corresponds to an embedding in the torus as depicted below:



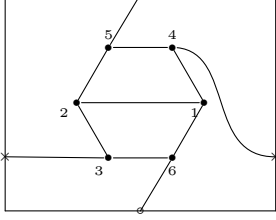
**Example 4.** Consider the case where one vertex has the opposite orientation relative to the two other vertices of the same set. For instance, the case where vertex 1 and 2 have positive orientation, (246) and (135) respectively, and the rest have negative orientation:

Vertex	1	2	3	4	5	6
Neighbors/Orientation	(246)	(135)	(264)	(153)	(264)	(153)

Using the rotation rule we obtain the following circuits,

- (1)  $(12) \rightarrow (23) \rightarrow (36) \rightarrow (61) \rightarrow (12)$ ;
- (2)  $(21) \rightarrow (14) \rightarrow (45) \rightarrow (52) \rightarrow (21)$ ;
- (3)  $(32) \rightarrow (25) \rightarrow (56) \rightarrow (63) \rightarrow (34) \rightarrow (41) \rightarrow (16) \rightarrow (65) \rightarrow (54) \rightarrow (43) \rightarrow (32)$ .

Notice that this time, we obtain two circuits of length four and a single one of length ten. Hence, the 18 sides available form three faces and the embedding is in a torus:



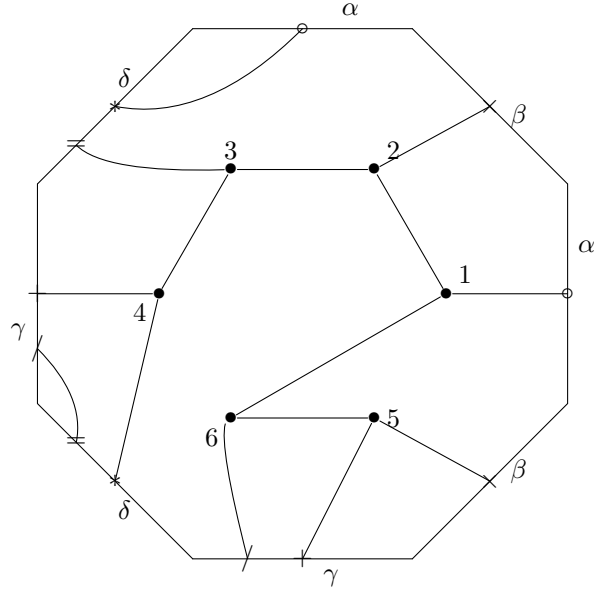
**Example 5.** Finally, consider the case where the orientation of all vertices in one set is the same while in the other set we have one vertex with the opposite orientation relative to the other two vertices. For instance, the case where all odd vertices have orientation (246) and vertex 2 has positive orientation as well, while the vertices 4 and 6 have orientation (153):

Vertex	1	2	3	4	5	6
Neighbors/Orientation	(246)	(135)	(246)	(153)	(246)	(153)

In this case we obtain, after using the described algorithm, only one circuit with 18 sides,

$$(12) \rightarrow (23) \rightarrow (34) \rightarrow (41) \rightarrow (16) \rightarrow (65) \rightarrow (52) \rightarrow (21) \rightarrow (14) \rightarrow (45) \rightarrow (56) \rightarrow \dots \\ \dots \rightarrow (63) \rightarrow (32) \rightarrow (25) \rightarrow (54) \rightarrow (43) \rightarrow (36) \rightarrow (61) \rightarrow (12)$$

where all sides are walked exactly once<sup>6</sup>. This configuration thus corresponds to an embedding in  $T^2 \# T^2$  which can be represented in a plane in a similar manner as the torus,



where the Greek letters denote the borders of the frame that have to be glued together in order to obtain a double torus and the symbols on them identify corresponding points.

Now, define the **value  $v$  of a set of vertices** as the modulus of the sum of orientations  $\pm 1$  of the vertices in that set, e.g. the value of  $\{1, 3, 5\}$  with orientation  $(-1 - 1 - 1)$  is  $|-3| = 3$ . The definition is such that, if the orientations of all vertices in a given set change, then the value remains invariant. For instance, one can achieve

<sup>6</sup>Notice that a circuit induced in this way may have repeated vertices and edges used in both directions, however, if the edge is repeated in the same direction the algorithm must stop, [Hartsfield and Ringel(2003)].

a change of orientation of all vertices in, say, the set  $R = \{1, 3, 5\}$ , e.g.  $(+1 + 1 - 1)$ , by an odd permutation of the vertices in  $S = \{2, 4, 6\}$  such that  $(+1 + 1 - 1) \mapsto (-1 - 1 + 1)$  but  $\mathbf{v}_{++-} = |+1| = |-1| = \mathbf{v}_{--+}$ . In fact, by permutations of the vertices in a set, one can construct all equivalent diagrams<sup>7</sup>, i.e. giving the same embedding, since these operations do not change the 2-cells of the embedding, it merely results in a permutation of the vertices in it.

Observe that the three cases in the examples above are the only cases possible if we consider only the *relative* orientation between vertices of the same set, in which case the value is either  $\mathbf{v} = 3$ , when all vertices have the same orientation, or  $\mathbf{v} = 1$ , when one of the vertices has the opposite orientation relative to the other two in the same set. The value of each set is independent of each other, hence, we have the cases where the pair of values are  $(3, 3)$ ,  $(3, 1)$ ,  $(1, 3)$  and  $(1, 1)$ . However, since  $K_{3,3}$  is symmetric under exchange of sets  $R \leftrightarrow S$  preserving the orientation, we can regard  $(1, 3)$  and  $(3, 1)$  as equivalent cases.

Another way of looking at this is to consider the partition of 18 in summands with some constraints. Each of the summands represents a circuit and their value represent the length of the circuit. The defining characteristics of the  $(3, 3)$ -bipartite graph do not allow the construction of circuits with an odd number of edges since this would mean that two vertices of the same set are adjacent. Thus, the partition of 18 cannot contain any odd numbers. It follows that the smallest possible circuit has length 4. Furthermore, the only number of summands in the partition can be 1 or 3 since they represent the regions of the embeddings. From these restrictions we conclude that the only partitions of 18 allowed are  $18$ ,  $6 + 6 + 6$ ,  $4 + 4 + 10$  and  $4 + 8 + 6$ . However, the latter partition is not realized. To see this, notice that it is not possible to construct a 4-circuit which is not a 4-cycle<sup>8</sup> since this would mean that either one edge is repeated, in which case the edge would need to have two loops at each vertex, or two edges are repeated, this is not possible since all vertices in the graph are 3-valent. Thus, the only possibility is to have a 4-cycle; this implies automatically the existence of another 4-cycle. To see this, notice that the 4-cycle has two vertices of each set, therefore, there are two more vertices available to construct the graph, one of each kind. The defining characteristics of  $K_{3,3}$  impose the constraint that these two vertices must be adjacent to each other and to the corresponding vertices in the original 4-cycle. This leaves no other possibility but to construct another 4-cycle, in contradiction to the partition  $4 + 8 + 6$ .

*Remark.* Observe the symmetry of the bipartite graph under permutation of its vertices reflected in the pair of values of the sets as well as in the partition of 18, thus, we may call the  $(3, 3)$  case “maximal symmetric”, the  $(1, 1)$  case “minimal symmetric” and the  $(3, 1) = (1, 3)$  case “asymmetric”. Therefore, we can think of this pair of values as the “degree of symmetry” of the graph.

From the examples 3, 4 and 5 we see that the partitions  $[6 + 6 + 6]$ ,  $[4 + 4 + 10]$  and  $18$  correspond to the pair of values  $(3, 3)$ ,  $(1, 1)$  and  $(1, 3)$  respectively. We say that the maximal (minimal) symmetric graph has a  $[6 + 6 + 6]$ -type ( $[4 + 4 + 10]$ -type) embedding and the asymmetric graph has a  $18$ -type embedding. Therefore we can say that the type of embedding is only dependent on the “degree of symmetry” of the graph given by the pair of values of the two sets  $R$  and  $S$ . In other words, the embeddings are topologically invariant under permutations acting on the sets of even and odd vertices. Odd permutations on one set, merely change the orientation of all vertices in the other set, in which case the value of the set is not affected. Even permutations on one set only affect the cyclic order of the orientations in that set, e.g. if we have an orientation  $(-1 + 1 + 1)$  of the set  $R$  an even permutation acting on  $R$  would only result in, say,  $(-1 + 1 + 1) \mapsto (+1 + 1 - 1)$ .

Thus, from all 64 possible configurations of the orientations of vertices in  $K_{3,3}$  only three of them induce inequivalent embeddings. If the pair of values is  $(3, 3)$ , then there are 4 equivalent configurations which induce a  $[6 + 6 + 6]$ -type embedding; either all 6 vertices have positive (or negative) orientation or 3 vertices from one set have positive (or negative) orientation while the vertices from the other set have opposite orientation. Therefore, we are left with  $64 - 4 = 60$  configurations; 36 from them belong to the case where the pair of value is  $(1, 1)$ . This is a  $[4 + 4 + 10]$ -type embedding, hence, one of the vertices on each set has the opposite orientation relative to the vertices from the set which belongs to, i.e. the sets have the orientations of the form  $(+1 - 1 - 1)$  and cyclic, or of the form  $(-1 + 1 + 1)$  and cyclic. Therefore, for each relative orientation there are 3 cases, which make  $3 \times 2 = 6$  for each set. The rest 24 of the configurations belong to the case where the pair of values is  $(3, 1) = (1, 3)$ . This gives an embedding in the double torus. There are 6 cases

<sup>7</sup>One has to consider the operation  $R \rightleftharpoons S$  as well.

<sup>8</sup>A **cycle** is a circuit which does not have any repeated edges in any direction.

where the value of a set is 1 and 2 cases where the value is 3, hence, for each of the pairs  $(1, 3)$  and  $(3, 1)$  there are 12 configurations to consider.

Finally, we can summarize the above discussion by the following statement,

**Fact 6.** *If the value of the two disjoint sets of vertices in  $K_{3,3}$  is unequal, then the cellular embedding of  $K_{3,3}$  corresponds to an embedding in  $T^2 \# T^2$ ; otherwise the only cellular embeddings of  $K_{3,3}$  are in the torus  $T^2$ , such that it is a  $[6 + 6 + 6]$ -type embedding for the  $(3, 3)$ -value or a  $[4 + 4 + 10]$ -type embedding for the  $(1, 1)$ -value.*

This result is important since it will allow us to extract information of the terms needed in the evaluation of non-planar spin networks to account for their topology<sup>9</sup>. The graph contains topological information about the surface in which it is cellular embeddable and we can use recoupling theory to extract that information. The reason for this is that we are considering only cellular embeddings and we use all the information contained in the graph (number of edges, vertices and their orientation) to build the surfaces by the Rotation Scheme. Hence, the information of the topology of the surface must be contained in the graph itself; in other words, by reducing the graph in the embedding, we receive a factor in the evaluation that reflects the information of the graph being non-planar. That is why it is important to consider only cellular embeddings, the faces are only 2-cells homeomorphic to discs with no information about the global topology. For instance, in the case of the  $K_4$  graph we have two cellular embeddings in the torus, one with an embedded 3- and another with an embedded 4-cycle as cells and both wrapping the two circles of the torus. Both cellular embeddings give, in fact, different evaluations, however only up to a constant involving powers of  $q$  (or  $A$ ). In fact, these spin networks are contained in  $K_{3,3}$ , in the sense that reducing the graph of  $K_{3,3}$  in the torus via Moussouris' algorithm leads to such diagrams. We will call the embedding of the tetrahedron in the torus with an embedded 3-cycle the *toroidal coupling coefficient*.

There are of course (non-cellular) embeddings of a graph in surfaces with higher genus, however, it is not the graph containing the information about the topology of the surface but the surface itself. If we consider a non-cellular embedding in the torus of the complete graph on 4 vertices, i.e. the tetrahedron, and we "cut" the surface along the edge of the graph wrapping the circle of the torus, we will get a surface which is not homeomorphic to a disc and which contains the information about the topology of the torus. Thus, the graph in that configuration has no information about a non-trivial topology.

Due to the classification of closed oriented surfaces we expect that the information extracted from the torus is sufficient to extend Moussouris' algorithm for the evaluation of planar spin networks to the non-planar case. We believe that by knowing the evaluation of the spin network corresponding to the torus we can use it to evaluate all spin networks with higher genus in terms of products of this evaluation. To evaluate these spin networks it would be necessary to arrange them such that it is possible to "cut" their components (using the generalized Wigner-Eckart theorem, cf. [Moussouris(1983)]) corresponding to each handle of the oriented surface and evaluate each torus separately, this would give a sum of products of toroidal symbols<sup>10</sup>.

### 3. THE EVALUATION OF NON-PLANAR SPIN NETWORKS

In this section we discuss Moussouris' Decomposition Theorem and present its algorithm for the evaluation of planar spin networks, which relates these to the Ponzano-Regge partition function. We then give an improved version of the theorem and apply this algorithm to the graph  $K_{3,3}$  in order to extract the information needed to extend it to non-planar spin networks, i.e. we give the explicit form of the toroidal phase factor for the  $q$ -deformed case and discuss what needs to be done to achieve a generalization of the mentioned theorem.

**3.1. The Decomposition Theorem.** In [Moussouris(1983)] J. P. Moussouris proved a theorem which relates the spin networks with the Ponzano-Regge theory by reducing a recoupling graph<sup>11</sup> to a sum of products of coupling coefficients. This reduction, known as the Decomposition Theorem, gives an evaluation of the spin network only dependent on the labeling of the graph, as in [Turaev and Viro(1992)] for a manifold with boundary.

First, we give the explicit form of the identities used in Moussouris' algorithm in the form of the following definition.

<sup>9</sup>We mean by the topology of a graph, the topology of the surface in which the graph is embedded.

<sup>10</sup>cf. Def. 10 on page 12.

<sup>11</sup>Recall that a recoupling graph of a group  $G$  is a labelled 3-valent graph representing a contraction of tensors of  $G$ . In the following, the term "recoupling graph" will denote such a graph together with an orientation of its vertices.



**Definition 7.** *Moussouris' algorithm* is defined to be the application of a finite number of the following operations,

- The crossing identity:

$$\begin{array}{c} b \\ \diagup \\ \bullet \\ \diagdown \\ a \end{array} \begin{array}{c} j \\ \text{---} \\ \bullet \end{array} \begin{array}{c} c \\ \diagup \\ \bullet \\ \diagdown \\ d \end{array} = \sum_i \alpha_i \begin{array}{c} b \\ \diagup \\ \bullet \\ \diagdown \\ a \end{array} \begin{array}{c} i \\ \text{---} \\ \bullet \end{array} \begin{array}{c} c \\ \diagup \\ \bullet \\ \diagdown \\ d \end{array}$$

where  $\alpha_i = \Delta_i \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}_q$ ;  $\Delta_i = (-1)^{2i} [2i + 1]$  and  $[*]$  denotes the quantum integer defined as

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

- The excision identity:

$$\begin{array}{c} c \\ \diagup \\ \bullet \\ \diagdown \\ b \end{array} \begin{array}{c} j \\ \text{---} \\ \bullet \end{array} \begin{array}{c} k \\ \text{---} \\ \bullet \end{array} \begin{array}{c} a \\ \diagup \\ \bullet \\ \diagdown \end{array} = \left\{ \begin{array}{ccc} a & b & c \\ j & k & l \end{array} \right\}_q \begin{array}{c} a \\ \diagup \\ \bullet \\ \diagdown \\ c \end{array}$$

There are two versions of the mentioned theorem which we will present and analyze in this section in order to understand how the expansion of the algorithm for evaluating non-planar spin networks could be done. The second version of the theorem, called network version, is more general than the first one since it does not assume the spin network to be planar, however, it assumes implicitly the existence of an embedded cycle for the recoupling graph  $F$  to be reduced. This implies that the embedding of the graph in some surface has at least two 2-cells since a cycle induces a region homeomorphic to a disc by using only one side of each edge in the Rotation Scheme. First, we present without proof the planar version of the theorem found in [Moussouris(1983)] and after a short discussion we arrive to the heart of this paper and give the improved network version and its proof.

**Theorem 8. Planar version of the Decomposition Theorem:**

Let  $F$  be a **planar** recoupling graph and  $D(F)$  its dual relative to a particular embedding in the sphere. Let  $C(F)$  be a combinatorial 3-manifold produced by dissecting  $D(F)$  with internal edges  $x_1, x_2, \dots, x_p$  into tetrahedra  $T_1, \dots, T_q$ . Then, the evaluation of the recoupling graph is given by the amplitude

$$\Psi(F) = \sum_{x_1, \dots, x_p} \prod_{j=1}^p [x_j] \prod_{k=1}^q [T_k]$$

where  $[x_j]$  is the loop-value of the edge  $x_j$  and the  $[T_k]$ 's are the coupling coefficient associated with the tetrahedra  $T_k$ .

The successive application of the Alexander moves, which correspond in the dual form to the elimination of a 3-cycle and the crossing identity, results in the introduction of sufficient internal edges to dissect the interior of  $D(F)$  into tetrahedra, giving a combinatorial 3-manifold  $C(F)$  with  $D(F)$  as its boundary. This decomposition process is non-unique, however, the Biedenharn-Elliott identity and the orthogonality of the  $6j$ -symbols ensures the equivalence of the decompositions, [Moussouris(1983)]. This is a special case of the procedure to obtain the invariant<sup>12</sup> described in [Turaev and Viro(1992)].

Moussouris does mention the importance of the orientation of the vertices in the evaluation of the graph, pointing out that considering the orientation of the vertices results in so-called phase factors, which can

<sup>12</sup>Notice that in the amplitude given above the theta-net factors are missing. This is due to the fact that in [Moussouris(1983)] the spin networks are normalized such that the theta-nets are evaluated to one.

be isolated as values of graphs with two vertices with the same orientation. However, in the proof of the network version of the theorem this consideration enters only in the first and second steps of the induction on the number of vertices  $V$  in  $F$ , disregarding the fact that the orientation of the vertices of a recoupling graph affects the embedding of it in a surface, which might be such that there is no embedded cycle at all. This would mean that the spin network cannot be reduced straightforward. We will discuss this case later. Moreover, in the proof it is also assumed implicitly that after reducing all embedded cycles the only diagram left is either a coupling coefficient or a toroidal phase factor (cf. section 3.2); in the latter case we can call such a recoupling graph a toroidal spin network since the phase factor left at the end of the reduction contains the information of the graph being embedded in the torus<sup>13</sup>.

As seen in example 5, the appearance of an embedded cycle is not always the case and there exist spin networks which are irreducible if we only consider the operations described in [Moussouris(1983)], thus, Moussouris' Decomposition Theorem is limited to planar and toroidal spin networks. Hence, it is necessary to rewrite the Decomposition Theorem in a more precise manner in order to consider the case where the spin network is at most toroidal, i.e. with toroidal phase factors, which are especially important for the  $q$ -deformed case.

We will now give an improved version of the theorem and its proof following [Moussouris(1983)]. We modified the network version of the theorem stated in Moussouris' Ph.D. thesis to account for the discussion above.

**Theorem 9. Decomposition Theorem:**

*A recoupling graph  $F$  of a compact semi-simple group  $G$ , which is at most toroidal, can always be evaluated as a sum of products of coupling coefficients of  $G$  and toroidal phase factors.*

*Proof.* First we discuss the existence of an embedded cycle in any (at most) toroidal spin network from a graph-theoretical perspective. The fact that the spin network is at most toroidal means that it is either planar or cellular embeddable in the torus, in the former case the number of faces is at least 3 because a 3-valent graph has the number of edges proportional to the number of vertices

$$e = \frac{3}{2}v$$

Thus,  $v - e + f = 2 \Rightarrow 2f - 4 = v$  and we need at least  $v = 2$  to have a closed spin network. The existence of more than one 2-cell in a planar graph implies automatically the existence of an embedded cycle because having an embedded circuit means that somewhere along the walk on the circuit one has to "change" the side of the walk to return to the edge repeated in the opposite direction. On a plane and for closed graphs this can only be done by a loop in the walk, which bounds a region on the plane, i.e. it contains at least one embedded cycle. In this case the graph corresponding to the spin network contains a bridge, thus is not (vertex) connected. However, as we discussed above, we can be sure of the existence of an embedded cycle, even in this special case. For the more general case of higher connected graphs we have in fact from Steinitz's theorem, stating that every 3-connected<sup>14</sup> planar graph corresponds to a convex polyhedron, that planar 3-connected spin networks contain only cycles.

On the other hand, if the graph is cellular embeddable in the torus then  $v - e + f = 0 \Rightarrow 2f = v$  thus if  $v \geq 4$  there are at least 2 faces. This is the case for a toroidal coupling coefficient, where the embedding of  $K_4$  is non-planar. It contains an embedded 3-cycle and an embedded 9-circuit, thus the circuits should be related to the global topology of the surface.<sup>15</sup> Now, consider the case where we have one 2-cell, then the graph represents a toroidal phase factor. To construct bigger 3-valent graphs cellular embeddable in the torus one has to introduce vertices and edges in such a way, that the resulting graph has an even number of vertices and embeddable cycles or circuits. The existence of an embedded cycle is already in the case of  $v = 4$  with the help of Moussouris' algorithm assured. Introducing two more vertices and an edge results in three possible types of graphs; two of which contain an embedded cycle and one that has two circuits. However, the latter one can be transformed (in the sense of spin networks) into a graph containing an embedded

<sup>13</sup>The coupling coefficient with a toroidal phase factor has as its cellular embedding exactly the one discussed at the end of the previous section.

<sup>14</sup>A  $k$ -connected graph is one which stays connected even if one removes any number of vertices smaller than  $k$ .

<sup>15</sup>The existence of a circuit for 3-connected graphs may imply non-planarity in general, but not the other way around, since the (6+6+6)-type embedding contains no circuit. We only consider this remark as an interesting point without pretending to state a general fact.

cycle<sup>16</sup>. Following these thoughts one concludes the existence of embedded cycles in more complex toroidal spin networks.

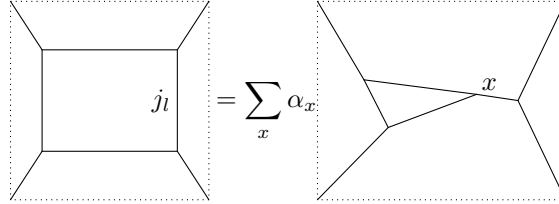
Having discussed the existence of embedded cycles in at most toroidal spin networks we turn now to the proof of the decomposition via Moussouris' algorithm. The proof is by induction on the number of vertices  $V$  in  $F$  and the size  $l$  of the smallest embedded cycle.

If  $V = 2$ , the recoupling graph is a toroidal phase factor or a theta-evaluation of a vertex. The case  $V = 3$  is not possible.

If  $V = 4$ , the recoupling graph is a (toroidal) coupling coefficient or two phases.

If  $V > 4$ , we look for the smallest embedded cycle in  $F$  and reduce it as follows, depending on its size  $l$ . A 2-cycle is reduced using Schur's identity. This results in a new graph containing  $V - 2$  vertices. A 3-cycle is eliminated by producing a single coupling coefficient by the Wigner-Eckart theorem. Alternatively, one can regard the so-called "crossing identity" described below to reduce the 3-cycle to a 2-cycle and apply Schur's identity. The resulting graph contains  $V - 2$  vertices.

For the case  $l > 3$  we have an embedded cycle with  $l$  edges labeled by  $j_1, j_2, \dots, j_l$ . This reduces to a  $(l - 1)$ -cycle by the crossing identity derived from using the Recoupling Theorem on the edge, say  $j_l$ , as depicted below. This operation results in a coupling coefficient multiplied by a recoupling graph in which the edge  $j_l$  is removed while a new "internal" edge  $x$  is introduced, coupling  $j_1$  to  $j_{l-1}$  and the other two edges, which were coupling to  $j_l$ , are also coupled to  $x$  and to each other. This resulting product is summed over the new edge  $x$  as in the Recoupling Theorem. The cycle is then reduced until  $l = 3$ .

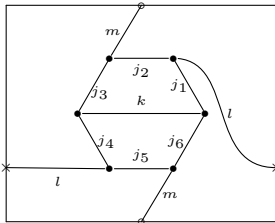


where  $\alpha_x = \Delta_x \left\{ \begin{matrix} a & b & x \\ c & d & j \end{matrix} \right\}$  are the so called coupling coefficients, a product of a  $6j$ -symbol and the "value" of the edge  $x$  give by  $\Delta_x = (-1)^{2x} [2x + 1]$ , with the brackets meaning that the integer  $2x + 1$  is taken to be a quantum integer for  $q \neq 1$ .

This process of vertex reduction is repeated until  $V = 4$  giving as a result a product of coupling coefficients and a phase factor summed over all internal variables.  $\square$

**3.2. The Evaluation of the Toroidal Spin Network  $K_{3,3}$ .** We will now apply the algorithm described in the above proof to specific spin networks of the  $[4 + 4 + 10]$ - and  $[6 + 6 + 6]$ -type embeddings of  $K_{3,3}$  on a torus, denoted by  $K_{3,3}^{(-1,+1)}$  and  $K_{3,3}^{(-3,+3)}$  respectively<sup>17</sup>. This will be done in order to extract the information about the topology encoded in the graph for the evaluation of non-planar spin networks.

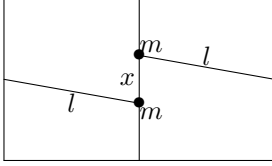
In the case of the embedding  $K_{3,3}^{(-1,+1)}$  we may start by applying the crossing identity to the common edge of the 4-cycles and then eliminating the two resulting 3-cycles by extracting two coupling coefficients.



<sup>16</sup>This is a very special case, which could also be considered as being two toroidal phase factors connected imposing a very strong condition for the spin network not to vanish.

<sup>17</sup>Observe that the labels  $(-1, +1)$  and  $(-3, +3)$  have the sign of the overall sum over the orientations of each set and not only the value of the set. The reason for this is that the evaluation depends on the configuration of the orientations on each set of  $K_{3,3}$ , as we will see later.

The result is a sum over a single internal edge  $x$  of a product of three  $6j$ -symbols weighted by a factor of  $(-1)^{2x}[2x+1]$ . These are, however, not all the factors since the diagram left encodes the information of the graph being embedded in a torus. This diagram, which we will call **toroidal phase factor**, can be represented in a torus as follows:



If we “project” this diagram to the plane by connecting the loose ends of the edges  $m$  and  $l$ , once we have disregarded the frame of the above diagram, we get a theta-net with these edges crossing. In order to get a more familiar theta-net, which can then be set to have the value of 1, we need to “twist” the edge  $x$ . This is done by following operation on a vertex<sup>18</sup>, [Carter et al.(1995)Carter, Flath, and Saito, Kauffman and Lins(1994)],

$$(3.1) \quad \begin{array}{c} j \\ | \\ a \quad \bullet \quad b \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = (-1)^{a+b-j} A^{2[a(a+1)+b(b+1)-j(j+1)]} \begin{array}{c} j \\ | \\ \bullet \\ \diagdown \quad \diagup \\ b \quad a \end{array},$$

where  $A = q^2$  is the deformation parameter of the underlying quantum group. The result of applying Moussouris algorithm and the above twisting rule is a sum of products of coupling coefficients as in the planar case, however, the non-planar nature of the graph is reflected in the “twist factor” given above, i.e. in the evaluation of the toroidal phase factor. Thus, we have

$$(3.2) \quad [K_{3,3}^{(-1,+1)}] = \sum_x \Delta_x \left\{ \begin{array}{ccc} j_3 & j_4 & k \\ j_6 & j_1 & x \end{array} \right\} \left\{ \begin{array}{ccc} m & j_5 & j_6 \\ j_4 & x & l \end{array} \right\} \left\{ \begin{array}{ccc} j_2 & l & j_1 \\ x & j_3 & m \end{array} \right\} (-1)^{l+m-x} A^{2[l(l+1)+m(m+1)-x(x+1)]}$$

where  $\Delta_x = (-1)^{2x}[2x+1]_q$  is the loop-evaluation. The squared brackets on the left hand side of the equation denote the evaluation of a graph in terms of coupling coefficients.

*Remark.* The equation (3.2) looks similar to the  $9j$ -symbol. However, considering the cases where  $A = \pm 1$ , we have an overall factor of  $(-1)^{l+m+x}$  which corresponds to one of the coupling coefficients having a vertex with the “wrong” orientation. This can be seen by expressing one of the  $6j$ -symbols where the labels  $x, l, m$  form an admissible triple in terms of  $3j$ -symbols and permuting the order of the labels in the corresponding  $3j$ -symbol by the following relation, [Edmonds(1996)],

$$(-1)^{j_1+j_2+j_3} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) = \left( \begin{array}{ccc} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{array} \right).$$

The resulting factor is exactly the one described in [Moussouris(1983), p. 65], i.e. a tetrahedron with one of the vertices having an orientation so that two edges cross. In fact, the diagram left after applying the crossing identity and eliminating only one of the two 3-cycle gives such a tetrahedron. Notice that if we disregard the information encoded in the orientation of the vertices, i.e. we set the phase factor equal to 1, we get the usual  $9j$ -symbol.

**Definition 10.** The *toroidal symbol* is defined in the following way

$$\left[ \begin{array}{ccc} j_1 & j_4 & j_7 \\ j_2 & j_5 & j_8 \\ j_3 & j_6 & j_9 \end{array} \right]_{A,+}^{(j_2,j_6)} := \sum_x \Delta_x \mathcal{A}_{j_2,j_6,x}^+ \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_6 & j_9 & x \end{array} \right\} \left\{ \begin{array}{ccc} j_4 & j_5 & j_6 \\ j_2 & x & j_8 \end{array} \right\} \left\{ \begin{array}{ccc} j_7 & j_8 & j_9 \\ x & j_1 & j_4 \end{array} \right\}$$

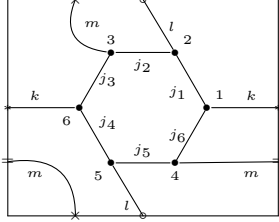
where  $\mathcal{A}_{j_2,j_6,x}^+ = (-1)^{j_2+j_6-x} A^{2[j_2(j_2+1)+j_6(j_6+1)-x(x+1)]}$  and we call the labels  $(j_2, j_6)$  the indices of the toroidal symbol.

<sup>18</sup>Notice that here we are presenting the case of a vertex with an over-crossing, however, this operation is defined for an under-crossing as well. In this case we exchange  $A \rightarrow A^{-1}$ . This amounts to the choice of orientation in the corresponding surface. Furthermore, observe that there is an arbitrary choice of the twist factor since one can change the sign in the exponent by performing some isotopies and using the fact that a curl in an edge  $a$  gives a factor of  $(-1)^{-2a} A^{\pm 4a(a+1)}$ , [Kauffman and Lins(1994)].

The symbol corresponding to the evaluation (3.2) is given by

$$(3.3) \quad [K_{3,3}^{(-1,+1)}] = \left[ \begin{array}{ccc} j_1 & j_6 & k \\ l & j_5 & j_4 \\ j_2 & m & j_3 \end{array} \right]_{A,+}^{(l,m)}.$$

Consider now the evaluation  $[K_{3,3}^{(-3,+3)}]$ . It is possible to reduce the  $[6+6+6]$ -type embedding to the  $[4+4+10]$ -type one by applying the crossing identity on two edges of the hexagonal figure shown in example 3 which belong to the same “exterior” 6-cycle, for instance the edges  $j_2 = (23)$  and  $j_6 = (14)$



From this procedure we get

$$(3.4) \quad [K_{3,3}^{(-3,+3)}] = \sum_{x,y} \Delta_x \Delta_y \left\{ \begin{array}{ccc} j_1 & j_2 & l \\ m & x & j_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_6 & k \\ m & y & j_5 \end{array} \right\} \left[ \begin{array}{ccc} j_5 & l & j_4 \\ y & m & k \\ j_1 & x & j_3 \end{array} \right]_{A,+}^{(x,y)}$$

where the first  $6j$ -symbol is the result of the crossing identity on the edge  $j_2 = (23)$  and the second one is the result of the same identity on the edge  $j_6 = (14)$ .

When reducing the spin network  $K_{3,3}^{(-3,+3)}$  by applying the algorithm on edges of different “exterior” 6-cycles, for instance on  $j_2$  and  $j_5$ , the reduction takes longer, however, again the Biedenharn-Elliott identity ensures that the evaluation is independent of the decomposition procedure.

At this point it is important to describe some properties of the toroidal symbol defined above. The symmetries of the (quantum)  $6j$ -symbols make the toroidal symbol invariant under reflection on either of the diagonals. Furthermore, if one exchanges two columns and two rows leaving one of the arguments on the anti-diagonal fixed, then the toroidal symbol stays invariant.<sup>19</sup> Moreover, there is an important new property that makes it possible to change the indices of the toroidal symbol and to exchange the arguments in the anti-diagonal. The following identity is due to a relation between sums over products of three quantum  $6j$ -symbols found in [Carter et al.(1995)Carter, Flath, and Saito]<sup>20</sup>,

$$(3.5) \quad \left[ \begin{array}{ccc} y & k & m \\ j_5 & j_4 & l \\ j_1 & j_3 & x \end{array} \right]_{A,+}^{(x,y)} = \mathcal{A}_{k,j_5,(j_1+m)}^+ \left[ \begin{array}{ccc} y & j_1 & j_5 \\ m & x & l \\ k & j_3 & j_4 \end{array} \right]_{A,+}^{(y,j_4)}.$$

Observe that the notation  $\mathcal{A}_{k,j_5,(j_1+m)}^+$  above does not mean that the exponent of  $A$  corresponding to the third index is  $-(j_1+m)((j_1+m)+1)$ , but rather  $-(j_1(j_1+1)+m(m+1))$ . Notice also that using this relation twice amounts to the identity, thus, this transformation can be regarded as a “(pseudo-)symmetry”.

If we compare  $[K_{3,3}^{(-1,+1)}]$  with the  $9j$ -symbol we may recognize the possibility to use the following relation between a  $9j$ -symbol (without twist factor) and  $6j$ -symbols, as in [Edmonds(1996)],

$$(3.6) \quad \sum_{\mu} (2\mu+1) \left\{ \begin{array}{ccc} j_{11} & j_{12} & \mu \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{array} \right\} \left\{ \begin{array}{ccc} j_{11} & j_{12} & \mu \\ j_{23} & j_{33} & \lambda \end{array} \right\} = (-1)^{2\lambda} \left\{ \begin{array}{ccc} j_{21} & j_{22} & j_{23} \\ j_{12} & \lambda & j_{32} \end{array} \right\} \left\{ \begin{array}{ccc} j_{31} & j_{32} & j_{33} \\ \lambda & j_{11} & j_{21} \end{array} \right\}$$

However, this relation does not account for the twist factor, hence, it is not possible to use straightforward.

<sup>19</sup>Notice that the arguments in the anti-diagonal are the ones paired with the argument of the three  $6j$ -symbols over which it is summed, i.e. they only appear once in the product of  $6j$ -symbols.

<sup>20</sup>The relation is given in the reference in a different form, namely, as a sum of products of three  $6j$ -symbols and a factor similar to the twist factor described above. We used the proof for the case  $A = \pm 1$  given in [Carter et al.(1995)Carter, Flath, and Saito] as a guide to reconstruct the relation in order to present it as a “(pseudo-)symmetry” of the toroidal symbol.

*Claim 11.* The identity corresponding to equation (3.6) is

$$(3.7) \quad \sum_{\mu} [2\mu + 1] \left[ \begin{array}{ccc} j_1 & j_4 & \mu \\ j_2 & j_5 & j_8 \\ j_3 & j_6 & j_9 \end{array} \right]_{A,+}^{(j_2, j_6)} \left\{ \begin{array}{ccc} j_1 & j_4 & \mu \\ j_8 & j_9 & \lambda \end{array} \right\} = \mathcal{A}_{j_2, j_6, \lambda}^+ \left\{ \begin{array}{ccc} j_2 & j_5 & j_8 \\ j_4 & \lambda & j_6 \end{array} \right\} \left\{ \begin{array}{ccc} j_3 & j_6 & j_9 \\ \lambda & j_1 & j_2 \end{array} \right\}$$

where  $[2\mu + 1]$  is a quantum integer corresponding to the loop-value of  $\mu$ .

*Proof.* The only term containing  $\mu$  in the expansion of the l.h.s. in term of  $6j$ -symbols is of the form

$$\sum_{\mu} [2x + 1][2\mu + 1] \left\{ \begin{array}{ccc} j_{11} & j_{12} & \mu \\ j_{23} & j_{33} & x \end{array} \right\} \left\{ \begin{array}{ccc} j_{11} & j_{12} & \mu \\ j_{23} & j_{33} & \lambda \end{array} \right\} = \delta_{x, \lambda}.$$

Thus, the only term left is the one on the r.h.s. □

*Remark.* Notice that (3.7) only holds if the toroidal symbol has that exact form, i.e.  $\mu$  must be in any counter-diagonal position. Labels in that position appear in the expansion equation (3.2) only in one  $6j$ -symbol, thus, the orthogonality of the  $6j$ -symbols may be used straightforward. Moreover, it is possible to transpose the toroidal symbol since this only changes the ordering of the admissible triples in the  $6j$ -symbols, i.e. it is possible to use the symmetry properties of the  $6j$ -symbols to achieve a transposition of the toroidal symbol. In the regular case without twist factor, the  $9j$ -symbols have some symmetries and this constraint does not appear. However, one can use the identity 3.5 on the preceding page, hence, even without having the symmetries needed it is possible to transform the toroidal symbol in (3.4) in order to bring it in a form suitable for the use of (3.7) to achieve a further simplification of  $[K_{3,3}^{(-3,+3)}]$ . Thus, assuming the triple  $(y, j_4, j_2)$  is admissible and using the following relation

$$\sum_x [2x + 1] \left\{ \begin{array}{ccc} j_1 & j_2 & l \\ m & x & j_3 \end{array} \right\} \left[ \begin{array}{ccc} y & j_1 & j_5 \\ m & x & l \\ k & j_3 & j_4 \end{array} \right]_{A,+}^{(y, j_4)} = \mathcal{A}_{y, j_4, j_2}^+ \left\{ \begin{array}{ccc} y & m & k \\ j_3 & j_4 & j_2 \end{array} \right\} \left\{ \begin{array}{ccc} j_2 & l & j_1 \\ j_5 & y & j_4 \end{array} \right\}$$

we can simplify (3.4) further.

Summarizing the discussion above we obtain the result that the  $[6 + 6 + 6]$ -type embedding of the  $(3, 3)$ -bipartite graph as a spin network has following evaluation

$$(3.8) \quad [K_{3,3}^{(-3,+3)}] = \mathcal{A}_{2k, (j_5+j_4), (j_1+j_2)}^+ \left[ \begin{array}{ccc} j_1 & k & j_6 \\ j_2 & j_3 & m \\ l & j_4 & j_5 \end{array} \right]_{(-)}^{(k, m)}.$$

Notice the change of sign of the toroidal symbol. This is due to the choice of the orientation. In other words, observe that the edges  $l$  and  $m$  wrap the torus in both examples above in fact in a different way, thus, this amounts to choosing different crossings and with it different signs in  $\mathcal{A}_{\dots}^{\pm}$ . Furthermore, observe that the idea of using the twisting rule to change the orientation of a vertex and with it its embedding before evaluating the given spin network is highly arbitrary and naive. The factor resulting out of this process is not the same as the factor gotten above as a result of evaluating the spin network under consideration via the extended algorithm and the identities above.

It is interesting to observe the fact that the proportionality constant reflects the possible maximal values that the spin label  $l$  could have if seen as a summand of the decomposition of the tensor product of  $j_5$  and  $j_4$  or  $j_1$  and  $j_2$  giving the admissibility conditions in the respective vertex, where the intertwiners  $j_5 \otimes j_4 \rightarrow l$  and  $j_1 \otimes j_2 \rightarrow l$  sit.

Notice that the above relation is not exactly the toroidal symbol in (3.3), however, it is proportional to some toroidal symbol up to a choice of orientation of the surface in which it is embedded, cf. footnote (18). From this we learn that the orientation of the vertices change the evaluation of the spin network. To see exactly how this happens further work is needed. For instance, how does any two spin networks with the same embedding relate to each other? Are they related via the transformation (3.5) above or some symmetries? However, the general form of the symbol remains and we can observe that the topology of the surface in which the diagram is embedded is reflected in the evaluation of the spin network. Nevertheless, it is important to observe that even considering a single type of embedding, say  $[4 + 4 + 10]$ , e.g. with a fixed value  $\mathfrak{v}$  of the set

of vertices, the toroidal symbols are not all equal for different orientations but same  $\mathbf{v}$ , although the difference might be only a proportionality factor changing the indices of the symbol, for instance,

$$[K_{3,3}^{(-1,-1)}]^{(j_4, j_6)} = \mathcal{A}_{j_4, j_6, (l+m)}^+ [K_{3,3}^{(+1,+1)}]^{(l,m)},$$

hence, to find out how many distinct toroidal symbols actually are, and the relation among them, some further work is needed in understanding the relation between different orientations of the vertices with the same resulting embedding.

Finally, we consider the embedding of  $K_{3,3}$  in the double torus. As mentioned before, this embedding has only one 2-cell, thus, is not possible to reduce it by means of Moussouris' algorithm. It is easy to see that using the crossing identity on this type of spin network would delete one edge but also introduce another in the same circuit, thus the total length of the circuit does not change and there is no reduction of the graph. In order to decompose this spin network, it would be necessary to change the orientation of a vertex by the "twisting" operation defined above. This would give an overall twist factor dependent on the edges involved. This operation is, however, highly arbitrary since, depending on the choice of the vertex to be twisted, one obtains either of the embeddings above or even the original embedding, thus, it is not a viable way to proceed. Nevertheless, since we were able to identify the toroidal phase factor with the handle of the torus and obtained (up to orientation of the surface) a symbol corresponding to this surface, one might ask if all spin networks embeddable in an orientable closed surface with genus  $> 0$  could be expressed as a sum of products of (quantum)  $6j$ - and toroidal symbols, one for each handle. It is not hard to imagine the existence of graphs with such evaluation.

At this point the embedding of the  $(3,3)$ -bipartite graph in the double torus is of great interest since it might be the missing link needed to generalize the Decomposition Theorem for spin networks with genus  $> 1$ . We could use the inverse operations of the ones used in Moussouris' algorithm on this embedding in order to introduce enough vertices and edges such that the resulting graph has two components, one on each handle. It would then be possible to separate the components using the generalized Wigner-Eckart theorem, [Moussouris(1983)]. This could help us to study the possibility of an evaluation of non-planar graphs as a sum of products of (quantum)  $6j$ - and toroidal symbols. In fact, a theorem of Battle, Harary, Kodama and Youngs states that the orientable genus of a graph is additive over its blocks, [Beineke and Wilson(1997)], hence, together with the classification of orientable closed surface, we might expect the above mentioned generalization.

It might be useful to consider the case where the complete graph on four vertices  $K_4$  is toroidal. In this case, the embedding has a 4-cycle that can be reduced to give a tetrahedron with a vertex having the wrong orientation. Remember that the  $K_{3,3}$  graph is a minor of the complete graph on five vertices, as the toroidal phase factor is a minor of  $K_4$  cellular embedded in a torus, hence, it might be the case that the embedding of  $K_{3,3}$  in the 2-torus is a "phase factor of higher genus". These speculative considerations are based on some properties of the graphs, however, more work would be needed in order to understand the existence, if any, of these relations.

However, one has to be careful in making a claim about generalizing the Decomposition Theorem based only on toroidal symbols and phase factors, since the existence of forbidden families of graphs prevent us to rush into such conclusions. Following theorem is the generalization of Kuratowski's theorem 1 on page 2 for graphs which are not embeddable in surfaces  $S_h$  of a given genus  $h$ , [Beineke and Wilson(1997)].

**Theorem 12.** *For any surface  $S$ , the family of homeomorphically minimal graphs that are not embeddable in  $S$ , called the minimal forbidden family  $\mathcal{F}(S)$ , is finite.*

*The minimal forbidden family  $\mathcal{F}(S)$  is define to be the set of graphs having the following three properties: (i) if the graph  $G \in \mathcal{F}(S)$ , then  $G$  is not embeddable in  $S$ ; (ii) if a graph  $G$  is not embeddable in  $S$ , then  $G$  contains a subgraph homeomorphic to some graph in  $\mathcal{F}(S)$ ; (iii)  $\forall G \in \mathcal{F}(S) : G$  is not homeomorphic to any subgraph of another graph in  $\mathcal{F}(S)$ .*

This theorem might lower our expectations of being able to express a given graph in terms of toroidal symbols since we expect other spin networks to be non-toroidal. For instance, there are more than 800 minimal forbidden graphs known for the torus, [Beineke and Wilson(1997)]. Thus, we need to analyze the general case in order to determine if the evaluation of all spin networks with genus  $> 1$  can be expressed as a sum of products of quantum  $6j$ - and toroidal symbols. Nevertheless, using the analogy of the role of  $6j$ -symbols in the expression of toroidal symbols the hope remains that the latter could be the factors needed to evaluate spin networks of higher genus.

## 4. CONCLUSION

The above analysis of the minimal non-planar spin network  $K_{3,3}$  shows that the topology of the spin network, i.e. the orientation of its vertices and with it the surface in which it is embeddable, is reflected in its evaluation. This affects the overall behavior of the coupling symbol and gives rise to new “symmetry” properties. Moreover, three types, or families, of minimal non-planar spin networks were found. Two of them are toroidal and related by a constant factor involving exponents of the deformation parameter of the underlying quantum group. The examples given above of these two families of networks are representative of each of them, but are not the only ones since every configuration of the orientation of its vertices might give rise to another toroidal symbol, or at least proportional to one of the above, even if they are of the same family. In order to understand how all of these toroidal symbols relate to each other a short analysis of the effect on different configurations of orientations for a given type is needed.

To account for the non-planar nature of the spin networks discussed an extension of Moussouris’s algorithm was done using the twisting identity of the Temperley-Lieb recoupling theory. Since only cellular embeddings were used, it was possible to identify the resulting factor in the evaluation with the handle of the torus in which the spin networks were embedded. It was, however, not possible to fully achieve the generalization of the theorem considered due to the inability of the algorithm to reduce unambiguously the third type of minimal spin network found, namely, the one embedded in a 2-torus with only one 2-cell. Instead some considerations about how to proceed to attempt a generalization of the theorem and the issues involved were discussed. Some hints about the possibility of such generalization given by the classification of closed orientable surfaces and the additive nature of the genus of a graph are present, however, one has to be careful to conclude a generalization based on these hints due to the existence of minimal forbidden families of graphs for a given surface of genus  $h$ . For this reason, one would need to consider the general case and extend the algorithm accordingly, maybe with the generalized Wigner-Eckart theorem for coupling graphs as given in [Moussouris(1983)], to account for arbitrary graphs of higher genus.

**Acknowledgements:** I am deeply grateful to John Barrett for having me in Nottingham and allowing me to gain an understanding of a broad spectrum of very interesting concepts. I also want to thank the University of Nottingham and the Erasmus Internship Programm for supporting me during my stay; thanks as well to the staff at the University of Munich for being so helpful and patient, especially Mrs. Schleiss and Mr. Emmer. Finally, I want to thank S. Hofmann for making it possible for me to be in Nottingham.

## REFERENCES

- [Barrett and Westbury(1996)] J.W. Barrett and B.W. Westbury. Invariants of piecewise-linear 3-manifolds. *Transactions of the American Mathematical Society*, 348(10):3997–4022, 1996. ISSN 0002-9947.
- [Beineke and Wilson(1997)] L.W. Beineke and R.J. Wilson. *Graph Connections: Relationship Between Graph Theory and Other Areas of Mathematics*. Clarendon Press, 1997.
- [Carter et al.(1995)Carter, Flath, and Saito] J.S. Carter, D.E. Flath, and M. Saito. *The classical and quantum 6j-symbols*, volume 43. Princeton Univ Pr, 1995.
- [Edmonds(1996)] A.R. Edmonds. *Angular momentum in quantum mechanics*. Princeton Univ Pr, 1996. ISBN 0691025894.
- [Hartsfield and Ringel(2003)] N. Hartsfield and G. Ringel. *Pearls in graph theory: a comprehensive introduction*. Dover Pubns, 2003.
- [Kauffman and Lins(1994)] L.H. Kauffman and S. Lins. *Temperley-Lieb recoupling theory and invariants of 3-manifolds*. Princeton Univ Pr, 1994. ISBN 0691036403.
- [Moussouris(1983)] J.P. Moussouris. Quantum models of space-time based on recoupling theory. 1983.
- [Ponzano and Regge(1969)] G. Ponzano and T. Regge. SEMICLASSICAL LIMIT OF RACA COEFFICIENTS. Technical report, Princeton Univ., NJ, 1969.
- [Regge(1961)] T. Regge. General relativity without coordinates. *Il Nuovo Cimento (1955-1965)*, 19(3):558–571, 1961.
- [Turaev and Viro(1992)] V.G. Turaev and O.Y. Viro. State sum invariants of 3-manifolds and quantum 6j-symbols. *Topology*, 31(4):865–902, 1992. ISSN 0040-9383.